Exercise 34

An **integral equation** is an equation that contains an unknown function $y(x)$ and an integral that involves $y(x)$. Solve the given integral equation. [*Hint:* Use an initial condition obtained from the integral equation.]

$$y(x) = 2 + \int_1^x \frac{dt}{ty(t)}, \quad x > 0$$

Solution

Right away we see that if we plug in $x = 1$ to the integral equation, we get

$$\begin{aligned} y(1) &= 2 + \int_1^1 \frac{dt}{ty(t)} \\ &= 2 + 0 \\ &= 2, \end{aligned}$$

so the initial condition is $y(1) = 2$. In order to solve for $y(x)$ by the method introduced in this section, differentiate both sides of the integral equation with respect to x.

$$\frac{d}{dx}y(x) = \underbrace{\frac{d}{dx}2}_{=\,0} + \frac{d}{dx}\int_1^x \frac{dt}{ty(t)}$$

By the fundamental theorem of calculus,

$$\frac{d}{dx}\int_1^x \frac{dt}{ty(t)} = \frac{1}{xy(x)},$$

so the differential equation we need to solve is the following.

$$\frac{dy}{dx} = \frac{1}{xy}$$

This is a separable equation, which means we can solve for $y(x)$ by bringing all terms with y to the left and all constants and terms with x to the right and then integrating both sides.

$$\begin{aligned} dy &= \frac{1}{xy}\,dx \\ y\,dy &= \frac{dx}{x} \\ \int y\,dy &= \int \frac{dx}{x} \\ \frac{1}{2}y^2 &= \ln|x| + C \\ y^2 &= 2\ln x + C_1 \\ |y| &= \sqrt{2\ln x + C_1} \end{aligned}$$

$$y(x) = \pm\sqrt{2\ln x + C_1}$$

Now we use the initial condition, $y(1) = 2$ to determine C_1.

$$\begin{aligned} y(1) = \pm\sqrt{C_1} &= 2 \\ C_1 &= 4 \end{aligned}$$

Even though both functions for y satisfy the differential equation, only the positive one satisfies the integral equation. Therefore,

$$y(x) = \sqrt{2 \ln x + 4}, \quad x \geq e^{-2}.$$

The domain is found by setting $2 \ln x + 4 \geq 0$.